*On a Certain General Class of Functional Equations.**

By W. Harold Wilson.

§1. *Introduction and General Considerations.*

Addition formulae of the general type

$$G[f(x+y), f(x), f(y)] = 0,$$

where G is a polynomial in its three arguments, play a prominent rôle in the theory of elliptic functions. A natural generalization of such formulae is

$$P[x, y, f(x), f(y), f(\alpha_1 x+\beta_1 y), \ldots, f(\alpha_n x+\beta_n y)] = 0, \tag{I}$$

where

(i) P denotes a polynomial in its $n+4$ arguments such that every argument involving f is explicitly present;

(ii) x and y are independent variables;

(iii) α_i and β_i, $i=1, 2, \ldots, n$, are given constants; † and,

(iv) $f(x)$ is an unknown single-valued function to be determined so that equation (I) shall be identically satisfied. ‡

Equation (I) *is said to be of order* n. *The degree* m *of* P *in the function* f *is said to be the degree of equation* (I).

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* Read before the American Mathematical Society (at Chicago), April 6, 1917.

† For the purposes of this paper it is convenient to carry certain hypotheses in regard to the α's and β's. A statement of these hypotheses is to be found below.

‡ A theorem of some interest in the general theory of these functional equations is that *every solution* $f(x)$ *of equation* (I) *is a solution of a similar equation in which* x *and* y *occur only in the arguments of the function* f. To prove this arrange P as a polynomial in x and y. The substitutions $x = s + k_i t$, $y = t$, where $k_0 = 0$ and

$$k_i \neq k_\lambda \pm \frac{\beta_h}{\alpha_h}, \quad k_i \neq k_\lambda + \frac{\beta_h}{\alpha_h} - \frac{\beta_j}{\alpha_j},$$

$\lambda = 0, 1, \ldots, i-1$, $h, j = 1, 2, \ldots, n$, transform (I) into equations similar to (I) such that the highest degree in s and t is the same for all of them. It is easily seen that a finite number of non-zero k's may be employed such that the variables s and t may be eliminated from these equations, in so far as they occur as coefficients, by Sylvester's dialytic method of elimination. The result of this elimination is an equation (II) which states that a polynomial Q in the function f has the value zero. The arguments of f are linear combinations of two independent variables s and t. Hence (II) is similar to (I), although in general its order and degree will differ from those of (I). Furthermore, if no two arguments of f in (I) are proportional, then no two arguments of f in (II) are proportional.

Cauchy * discussed two special cases of (I) and two related equations, namely:

$$f(x+y)=f(x)+f(y), \quad f(x+y)=f(x)f(y),$$
$$f(xy)=f(x)+f(y), \quad f(xy)=f(x)f(y).$$

One or the other of the first two of these equations has since been treated † by Darboux, E. B. Wilson, Vallée Poussin, Schimmack and Hamel. Carmichael ‡ has given a generalization of the Cauchy equations while Jensen § has discussed several applications of them. Cauchy || has treated the equation

$$\phi(x+y)+\phi(x-y)=2\phi(x)\phi(y).$$

Carmichael ¶ has considered the equations

$$h(x+y)h(x-y)=h^2(x)+h^2(y)-c^2, \quad g(x+y)g(x-y)=g^2(x)-g^2(y).$$

Van Vleck and H'Doubler ** have discussed the equation

$$\psi(x+y)\psi(x-y)=[\psi(x)\psi(y)]^2.$$

Other related equations have been considered by several writers, and systems of functional equations have also been treated.

It seems that no systematic account of a general theory for equations of the form (I) has ever been undertaken. This paper is designed to contribute to such an account. The equations considered in the principal part (§§ 2 to 8) of the paper are linear homogeneous equations with constant coefficients. They may be written in the form

$$\sum_{i=1}^{n} \gamma_i f(\alpha_i x+\beta_i y)+\gamma_{n+1}f(x)+\gamma_{n+2}f(y)=0. \tag{1}$$

It will be shown that if some α's and β's having different subscripts are zero and no ratio α_i/β_i of non-zero α's and β's is distinct from all the remaining ratios, the equation is exceptional. The exceptional case receives mention only in §§ 6 and 11. There is no loss of generality in assuming that no α is zero in the non-exceptional case. For convenience in exposition the hypothesis will be carried in the text that in addition to no α being zero, no β is zero, and no two ratios α_i/β_i are equal. The additional argumentation for the remaining non-exceptional cases is supplied in footnotes.

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* *Cours d'Analyse* (1821), Chapter 5. Cauchy treated the last two equations by transforming them into the first two equations. It is obvious that similar transformations may be applied to reduce more general equations to the form of those considered in this paper.

† Darboux, *Mathematische Annalen*, Vol. XVII (1880), p. 56. E. B. Wilson, *Annals of Mathematics*, Vol. I, Ser. 2 (1899), p. 47. Vallée Poussin, *Cours d'Analyse infinitésimale* (1903), p. 30. Schimmack, *Nova Acta*, Vol. XC, p. 5. Hamel, *Mathematische Annalen*, Vol. LX (1905), p. 459.

‡ *American Mathematical Monthly*, Vol. XVIII (1911), p. 198.

§ *Tidsskrift for Mathematik*, Vol. II, Ser. 4 (1878), p. 149.

|| *Cours d'Analyse* (1821), p. 114.

¶ *American Mathematical Monthly*, Vol. XVI (1909), p. 180.

** *Transactions of the American Mathematical Society*, Vol. XVII (1916), p. 9.

A normal equation of order n is derived (in § 2) which is satisfied by every solution of any non-exceptional equation (1) of order n. This normal equation forms a foundation upon which the entire development of the theory of equation (1) is based. Any normal solution may be uniquely determined at each point of a dense set covering the complex plane if it is given at the vertices of a certain triangular network (§§ 3, 4). It is shown in § 5 that *the normal solution analytic in the neighborhood of the point zero of the complex plane is an arbitrary polynomial in x of degree n.* It is also shown that *the normal solution continuous in the neighborhood of the point zero of the complex plane is an arbitrary polynomial in u and v of degree n where u and v are real and $x=u+v\sqrt{-1}$.* The normal solution analytic along any line in the finite complex plane is also an arbitrary polynomial in x of degree n and the normal solution continuous along any line in the finite complex plane is an arbitrary polynomial in u of degree n if the line is not parallel to the axis of imaginaries and an arbitrary polynomial in v of degree n if the line is not parallel to the axis of reals. The analytic and continuous solutions of (1) are found (§ 6) from the normal solution. Examples are exhibited in § 6 which show that equations of type (1) may have non-trivial continuous solutions, but no non-trivial analytic solutions, while other examples show that equations of type (1) may have analytic solutions which are also the most general continuous solutions. A converse theorem is briefly considered in § 7. It is shown (in § 8) that *if a function $f(x)$ satisfying an equation of type (1) has a point of discontinuity in the finite complex plane [or on any line in the finite complex plane] it has a point of discontinuity in every region [interval] of the plane [line], however small.*

Equation (1) is employed (§ 9) to solve certain equations of the type

$$\sum_{i=1}^{n}\phi_i(x, y)f(\alpha_i x+\beta_i y)+\phi_{n+1}(x, y)f(x)+\phi_{n+2}(x, y)f(y)+\phi_{n+3}(x, y)=0,$$

where the ϕ's are known functions. Equation (1) is also employed (§ 10) to find all analytic solutions, and in some cases, the continuous solutions of binomial equations of the type

$$\prod_{i=1}^{k}[f(\alpha_i x+\beta_i y)]^{\gamma_i}=C\prod_{i=k+1}^{n+1}[f(\alpha_i x+\beta_i y)]^{\gamma_i}[f(y)]^{\gamma_{n+2}},$$

where no α is zero, C is a constant and the γ's are constants of which the real parts are positive. Pexider* used the first Cauchy equation to solve

$$f(x)+\phi(y)=\psi(x+y).$$

In § 11 the method of obtaining the solutions of (1) is used to solve the equation

$$\sum_{i=1}^{n}\gamma_i f_i(\alpha_i x+\beta_i y)+\gamma_{n+1}f_{n+1}(x)+\gamma_{n+2}f_{n+2}(y)=0,$$

* *Monatshefte für Mathematik und Physik*, Vol. XIV (1903), p. 293.

of which the equation considered by Pexider is a special case. It is proved that when no two arguments of f in the foregoing equation are proportional, each continuous solution f is a polynomial of degree not greater than n.

§ 2. *Reduction to a Normal Form.*

The solution $f(x)$ of the general n-th order equation

$$\sum_{i=1}^{n}\gamma_i f(\alpha_i x+\beta_i y)+\gamma_{n+1}f(x)+\gamma_{n+2}f(y)=0, \tag{1}$$

is contained in that of an equation of the same form and same order [equation (6) below] in which each α, β, γ is a given integer. The derivation of (6) from (1) is accomplished by elimination.

If (1) is subtracted from the equation derived from (1) by replacing y by $y+t_{n+1}$, the result is

$$\sum_{i=1}^{n}\gamma_i[f(\alpha_i x+\beta_i y+\beta_i t_{n+1})-f(\alpha_i x+\beta_i y)]+\gamma_{n+2}[f(y+t_{n+1})-f(y)]=0. \tag{2}$$

If (2) is subtracted from the equation derived from (2) by replacing x by $x-\beta_1 t_1$ and y by $y+\alpha_1 t_1$, the result is

$$\begin{aligned}\sum_{i=2}^{n}\gamma_i[f(\alpha_i x+\beta_i y+\Delta_{1i}t_1+\beta_i t_{n+1})-f(\alpha_i x+\beta_i y+\Delta_{1i}t_1)\\ -f(\alpha_i x+\beta_i y+\beta_i t_{n+1})+f(\alpha_i x+\beta_i y)]\\ +\gamma_{n+2}[f(y+\alpha_1 t_1+t_{n+1})-f(y+\alpha_1 t_1)-f(y+t_{n+1})+f(y)]=0,\end{aligned} \tag{3}$$

where

$$\Delta_{ji}=\alpha_j\beta_i-\alpha_i\beta_j.$$

It is easily seen that this is true because the given substitutions leave $\alpha_1 x+\beta_1 y$ unchanged, but replace $\alpha_i x+\beta_i y$, $i\neq 1$, by

$$\alpha_i x+\beta_i y+\alpha_1\beta_i t_1-\alpha_i\beta_1 t_1=\alpha_i x+\beta_i y+\Delta_{1i}t_1.$$

In general, if an equation (a) results after such eliminations, then each argument of f in (a) is a linear expression in x, y and certain t's, the subscripts of the t's corresponding to those in the terms eliminated. The substitution of

$$x-\beta_j t_j \text{ for } x \quad\text{and}\quad y+\alpha_j t_j \text{ for } y \tag{4}$$

gives rise to an equation (b) which differs from (a) by having each $\alpha_i x+\beta_i y$ of (a) replaced by $\alpha_i x+\beta_i y+\Delta_{ji}t_j$. Since $\Delta_{jj}=0$ and since (4) does not affect the t's that are found in (a), it follows that the difference formed by subtracting (a) from (b) contains no term for which i is equal to the fixed integer j. Since n is finite these eliminations may be continued until only terms having the coefficients $\pm\gamma_{n+2}$ remain. Moreover, the order of elimination of the terms for which $i=1, 2, \ldots, n$, is immaterial.

It is obvious that the equation resulting from these eliminations is linear and homogeneous. Since $\gamma_{n+2} \neq 0$ by the assumption that (1) is of order n, the equation may be simplified by dividing by γ_{n+2}. It is easily seen that there are $(n+1)!/k!(n+1-k)!$ distinct terms in which the coefficient of f is $(-1)^k$, and in which the argument of f is obtained by omitting k terms after the first from $y+\alpha_1 t_1+\alpha_2 t_2+\ldots+\alpha_n t_n+t_{n+1}$. Moreover, no other such terms are possible. This is true for $k=0, 1, \ldots, n+1$. The substitutions t_1 for $\alpha_1 t_1$, t_2 for $\alpha_2 t_2$, $\ldots$, t_n for $\alpha_n t_n$, t_{n+2} for y, serve to completely determine an equation independent of the original α's, β's and γ's. If Σ_k denotes the sum of the $(n+1)!/k!(n+1-k)!$ terms in which the arguments are formed by omitting k t's from $\Sigma_{i=1}^{n+1} t_i$, then the equation which is satisfied by $f(x)$ is

$$\Sigma_0-\Sigma_1+\Sigma_2-\ldots+(-1)^n\Sigma_n+(-1)^{n+1}\Sigma_{n+1}=0. \tag{5}$$

Equation (5) involves $n+2$ independent variables. In order to obtain a normalized equation having the same form and order as (1), let $t_1=t_2=\ldots=t_{n+1}=x$ and $t_{n+2}=y$, whence (5) becomes*

$$f[(n+1)x+y]+\ldots+(-1)^{n+1-k}\frac{(n+1)!}{k!(n+1-k)!}f(kx+y)$$
$$+\ldots+(-1)^n f(x+y)+(-1)^{n+1}f(y)=0. \tag{6}$$

From the foregoing considerations we see that *every solution $f(x)$ of equation* (1) *is a solution of equation* (5) *and of the normal equation* (6).

While the solutions $f(x)$ of (1) are included among those of (6), it is not necessarily true that the solutions of (6) are included in those of (1). The following example suffices to prove this statement. Equation (6) for $n=2$ is

$$f(3x+y)-3f(2x+y)+3f(x+y)-f(y)=0. \tag{7}$$

As will be shown in § 5, the most general continuous solution of (7) over the finite complex x-plane is

$$f(x)=au^2+buv+cv^2+du+ev+f, \tag{8}$$

where $x=u+v\sqrt{-1}$ (u and v real) and a, b, c, d, e and f are arbitrary constants. From the above considerations we see that the solution of any second order equation of form (1) is included among those of (8). However, substitution shows that for the equation

$$f(2x+y)-2f(x+y)-2f(x)+f(y)=0$$

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* The results in (5) and (6) can be rendered more precise in special cases. If $\beta_j/a_j=\beta_h/a_h$, $\Delta_{jh}=0$ and the substitution of $x-\beta_j t_j$ for x and $y+a_j t_j$ for y leaves $a_h x+\beta_h y$, as well as $a_j x+\beta_j y$, unchanged. Hence the elimination of terms for which $i=j$ also eliminates all terms for which $i=h$ where h is any value for which $\Delta_{jh}=0$. But when $\Delta_{jh}=0$, $\beta_j/a_j=\beta_h/a_h$, and therefore this proportionality is a necessary and sufficient condition for the simultaneous elimination of terms for more than one value of i. Hence *the solution of any m-th order equation* (1) *in which no a is zero, is contained in the solution of equation* (5) *or of equation* (6) *where n is the number of distinct ratios β_i/a_i in* (1).

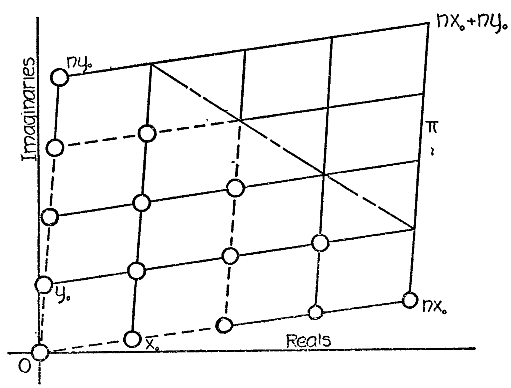
we have $d=e=f=0$. In fact, such an equation need have no other than the trivial solution $f(x)\equiv 0$ as is shown by the equation

$$f(2x+y)-f(x+y)-2f(x)+f(y)=0.$$

§ 3. *Determination of the Normal Solution at the Vertices of a Network.*

The function $f(x)$ to be considered in this section and in §§ 4, 5 is the solution of the normal equation developed in § 2. It is to be observed that if f is given for the arguments ky_0, x_0+ky_0, $2x_0+ky_0$, $\ldots$, nx_0+ky_0, then f is known for the argument $(n+1)x_0+ky_0$ by (6). By putting $y=jx_0+ky_0$ for successive values 1, 2, $\ldots$, of j, f is determined by (6) for any argument mx_0+ky_0, where m is any positive integer. Similarly, by successively giving to j the values -1, -2, $\ldots$, f is determined by (6) for any argument mx_0+ky_0, where m is a negative integer. By interchanging x and y in (6) it is easily seen that f is known for all arguments mx_0+ky_0, where k is any integer or zero, if it is given for the arguments mx_0, mx_0+y_0, mx_0+2y_0, $\ldots$, mx_0+ny_0. Hence, if f is given for the arguments mx_0+ky_0, $m, k=0, 1, \ldots, n$, then it is known by (6) for all arguments mx_0+ky_0, where m is any integer whatever and $k=0, 1, \ldots, n$, and finally, it is known by (6) for all arguments mx_0+ky_0, where m and k are any integers whatever.

It will now be proved that if f is known for the arguments mx_0+ky_0, where m and k are zero or positive integers and $m+k<n+1$, then it is also known for the arguments mx_0+ky_0, where $m, k=0, 1, \ldots, n$. The figure illustrates the case for which $n=4$. The points designated by the small circles



represent the arguments for which f is supposed known. The parallelogram formed by the lines joining the points 0, nx_0, nx_0+ny_0, and ny_0 will be denoted by π. The parallelogram π contains, on and within its boundary, all the points mx_0+ky_0, $m, k=0, 1, \ldots, n$, and no other points as vertices. It is required, therefore, to find f at the vertices of π not designated by small circles.

Equation (5) furnishes a simple means of solution of the present problem. Let $t_1=t_2=\ldots=t_q=x_0$, $t_{q+1}=\ldots=t_{n+1}=y_0$ and $t_{n+2}=0$. By these substitutions every argument in (5) is of the form mx_0+ky_0 where m and k are zero or positive integers and $m+k<n+1$, except in the first term in which $m+k=n+1$. Moreover, $m\leqq q$ and $k\leqq n+1-q$; hence f is determined for the argument $qx_0+(n+1-q)y_0$ in terms of linear combinations of its values for arguments represented by vertices within or on the boundary of the parallelogram bounded by lines joining the four points 0, qx_0, $qx_0+(n+1-q)y_0$, $(n+1-q)y_0$, (represented in the figure by dotted lines for $q=2$). By giving q the values 1, 2, ..., n, in succession, f is determined at each vertex of π which lies on the line joining the points x_0+ny_0 and nx_0+y_0 (the dot-and-dash line in the figure). It is to be noticed that each determination is made from a linear equation in one unknown with unit coefficient. Hence, the functional values so found are unique.

Having determined f for the arguments mx_0+ky_0 in π for which $m+k=n+1$, it is easy to determine f at the vertices for which $m+k=n+2$. In (5) let $t_1=t_2=\ldots=t_q=x_0$ and $t_{q+1}=\ldots=t_{n+2}=y_0$. Thus f is determined for the argument $qx_0+(n+2-q)y_0$ in terms of arguments represented by vertices within and on the boundary of the parallelogram formed by the four lines joining the points 0, qx_0, $qx_0+(n+2-q)y_0$, $(n+2-q)y_0$. This determination is also made by means of a linear equation involving but one unknown with the coefficient $+1$. Hence, no indetermination can be introduced. By giving q the successive values 2, 3, ..., n, f is determined for all arguments in π which lie on the line joining the points $2x_0+ny_0$ and nx_0+2y_0.

Proceeding in this manner f is determined for all the remaining arguments of π by giving t_{n+2} the values $2y_0$, $3y_0$, ..., $(n-1)y_0$ successively. For each value hy_0 of t_{n+2}, f must be determined for all arguments

$$qx_0+(n+h+1-q)y_0,\quad q=h+1,\quad h+2,\ \ldots,\ n,$$

before giving t_{n+2} the value $(h+1)y_0$. As before, each determination is made by a linear equation in one unknown with coefficient $+1$. Hence, f is uniquely determined at all the vertices of π, and we have the following result:

Every solution $f(x)$ of the normal equation (6) is known for the points mx_0+ky_0 where m and k are any integers of zero if it is given at the points mx_0+ky_0 for which m and k are positive integers or zero and $m+k<n+1$.

§ 4. *Determination of the Normal Solution at a Dense Set of Points.*

It will now be shown that if the solution of the normal equation is known at all points mx_0+ky_0, where m and k are integers, then it may be found for

$\frac{1}{2}mx_0+\frac{1}{2}ky_0$, and finally for $2^{-s}mx_0+2^{-s}ky_0$ for all integers m, k and s. If f is found for certain appropriate linear combinations of $2^{-s}x_0$ and $2^{-s}y_0$, then it is evident that $2^{-s}x_0$ and $2^{-s}y_0$ may be regarded as were x_0 and y_0 in § 3. Hence, if it is most convenient to determine f at $2^{-s}mx_0+2^{-s}ky_0$, $m, k=0, 1, \ldots, 2n$, it is also sufficient, in view of the argument of § 3, for f will then be known at these points for all integers m and k. Furthermore, if it is shown that f is known at $2^{-1}mx_0+2^{-1}ky_0$ for all integers m and k when it is known at mx_0+ky_0, then by regarding $2^{1-s}x_0$ and $2^{1-s}y_0$ as x_0 and y_0 for the successive values $s=2, 3, \ldots$, it is seen that f is known at $2^{-s}mx_0+2^{-s}ky_0$ for all integers m, k and s. Therefore, since the points $2^{-s}mx_0+2^{-s}ky_0$, m, k and s any integers whatever, form a dense set, it is sufficient to show that f is known at $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=0, 1, \ldots, 2n$, to show that f is known at all points of a dense set.

Supposing f known at $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=0, 2, \ldots, 2n$, it is only required to learn f at $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=1, 3, \ldots, 2n-1$. Replacing x by $\frac{1}{2}x_0$ and y by $\frac{1}{2}hx_0+ky_0$ in (6), n equations are obtained by giving h the values $n-1, \ldots, 1, 0$. The equations are linear in f. Alternate terms involve the arguments $\frac{1}{2}mx_0+ky_0$, $m=1, 3, \ldots, 2n-1$, for which f is unknown. The n equations may be considered as linear equations in n unknowns. It is easily seen that the determinant of the coefficients of the unknowns is $(-1)^{\nu}\Delta_n$, where ν is $n/2$ or $(n+1)/2$ according as n is even or odd, and Δ_n is a determinant of binomial coefficients such that the element in the i-th row and j-th column is

$$(n+1)!/(2j-i)!(n-2j+i+1)!,$$

unless $2j<i$ or $2j>n+i+1$, in which case it is zero. Subtracting the $(i+1)$-th row from the i-th row, i having the values $1, 2, \ldots, n-1$, in order, the element in the i-th row and j-th column, $i\neq n$, becomes

$$(n-4j+2i+2)\ (n+1)!/(2j-i)!(n-2j+i+2)!,$$

unless $2j<i$ or $2j>n+i+2$, in which case it is zero. Adding to the j-th column the sum of the preceding columns, giving j the values $n, n-1, \ldots, 2$ in order, Δ_n assumes a form in which the element in the i-th row and j-th column, $i\neq n$, is

$$S=\sum_{h=1}^{j}(n-4h+2i+2)(n+1)!/(2h-i)!(n-2h+i+2)!.$$

If for any given value of j this sum is $n!/(2j-i)!(n-2j+i)!$, unless $2j<i$ or $2j>n+i$, in which case it is zero, it is easily shown that for $j+1$ it is $n!/(2j-i+2)!(n-2j+i-2)!$, unless $2j<i-2$ or $2j>n+i-2$, in which case it is zero. For $2j-i=0$ or $2j-i=1$ the value of S is merely the first non-vanishing term, that is, 1 or n, respectively. Since $n!/(2j-i)!(n-2j+i)!$ is

equal to 1 for $2j-i=0$, and equal to n for $2j-i=1$, it follows by induction that S has the value
$$S=(n)!/(2j-i)!(n-2j+i)!,$$
unless $2j<i$ or $2j>n+i$, in which case it is zero. For the n-th column the value of S is always zero for $i\neq n$ since $2j>n+i$. For $i=n$, it is clear that the elements are affected only by the process of addition, and since every alternate term of $(1+1)^{n+1}$ is involved, the last term in the n-th column is 2^n. Hence, Δ_n has been so transformed that the principal $(n-1)$-rowed minor found by deleting the last row and last column of Δ_n is Δ_{n-1}, and the last column consists exclusively of zeros with the single exception of the element 2^n in the n-th row. Therefore $\Delta_n=2^n\Delta_{n-1}$. It is evident that $\Delta_2=2^{1+2}$ (Δ_1 is trivially 2). Therefore $\Delta_n=2^{1+2+\cdots+n}=2^{\frac{1}{2}n(n+1)}$ which is distinct from zero for all finite values of n. Since $(-1)^\nu\Delta_n\neq 0$, it follows immediately that f is uniquely determined at the points $\frac{1}{2}mx_0+ky_0$, $m=1, 3, \ldots, 2n-1$, and hence for $m=1, 2, \ldots, 2n$.

Having determined f at the points $\frac{1}{2}mx_0+ky_0$, $m=1, 2, \ldots, 2n$, it is only necessary to let $x=\frac{1}{2}y_0$ and $y=\frac{1}{2}hy_0+\frac{1}{2}mx_0$ in (6), letting h take the values $n-1, \ldots, 1, 0$, to determine f at the points $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=0, 1, \ldots, 2n$, for $(-1)^\nu\Delta_n$ is again the determinant of the coefficients of the unknown terms.

Combining the results of §§ 3 and 4, it may be stated that *f is determined at each point of the dense set $2^{-s}mx_0+2^{-s}ky_0$ if it is known at the points mx_0+ky_0 for which m and k are positive integers or zero and $m+k<n+1$.*

§ 5. *Solutions of the Normal Equation.*

Equation (6) may be written in the form
$$\sum_{j=0}^{n+1}(-1)^j\frac{(n+1)!}{j!(n-j+1)!}f[(ju+s)+\sqrt{-1}(jv+t)]=0,$$
where $x=u+v\sqrt{-1}$, $y=s+t\sqrt{-1}$, and u, s, v, t are real. Let us seek a solution $f(x)$ of this equation in the form
$$f(x)=\sum_{h=0}^{N}\sum_{q=0}^{h}c_{qh}u^qv^{h-q}, \tag{9}$$
in which the coefficients c_{qh} are constants. Putting this value of $f(x)$ in (6) we have
$$\sum_{j=0}^{n+1}(-1)^j\frac{(n+1)!}{j!(n-j+1)!}\sum_{h=0}^{N}\sum_{q=0}^{h}c_{qh}(ju+s)^q(jv+t)^{h-q}=0.$$
The terms involving $u^{q-p}s^pv^{h-q-r}t^r$ are
$$\sum_{j=0}^{n+1}\left[(-1)^j\frac{(n+1)!}{j!(n-j+1)!}j^{h-p-r}\right]c_{qh}\frac{q!}{p!(q-p)!}\frac{(h-q)!}{r!(h-q-r)!}u^{q-p}s^pv^{h-q-r}t^r.$$

Since no term outside of the brackets involves j, it is evident that the given expression vanishes, for non-zero c_{qh}, u, s, v and t, when and only when

$$B_{qh} \equiv \sum_{j=0}^{n+1} \left[(-1)^j \frac{(n+1)!}{j!(n-j+1)!} j^{h-p-r} \right]$$

vanishes. Except for sign the quantity B_{qh} is the $(n+1)$-th difference of x^{h-p-r} for $x=0$. The degree of each difference is one less than that of the preceding difference. Therefore B_{qh} is zero when $h-p-r \leqq n$. The particular value h obtained from $h-p-r$ by putting $p=r=0$ must be included in the discussion of B_{qh}. Since N was taken as the largest value of h, it follows that $f(x)$, as given by (9), satisfies (6) if $N=n$. Moreover, the value of c_{qh} is arbitrary.

Equation (9) shows that there are $\Sigma_{h=0}^{n}(h+1) = \frac{1}{2}(n+1)(n+2)$ arbitrary constants c_{qh} which may be assigned at will. By §§ 3 and 4 it has been shown that when f is known at the $1+2+\ldots\ldots+(n+1) = \frac{1}{2}(n+1)(n+2)$ points mx_0+ky_0, m and k positive integers or zero and $m+k<n+1$, then it is known over a dense set of points covering the entire finite plane provided x_0 and y_0 are not collinear with the point zero. Since $f(x)$, as given by (9), is continuous, it is only necessary to prove that each c_{qh} is uniquely determined by assigning f at the given points mx_0+ky_0 to know that $f(x)$ is the most general continuous solution of (6) over the finite complex plane. This can be done by direct substitution,* but inasmuch as the determinant so formed is unwieldy, it is more easily accomplished by observing that the properties sought for any desired oblique network are readily deduced by a projective transformation from similar properties of a square array, on the axes of reals and imaginaries with the units 1 and i, provided only that x_0 and y_0 do not lie on the same straight line through the zero-point. Confining attention to the rectangular array mentioned, write

$$f(x) = \sum_{h=0}^{n} \sum_{q=0}^{h} A_{qh} u^{(q)} v^{(h-q)}$$

where

$$u^{(q)} = u(u-1)\ldots\ldots(u-q+1).$$

For any integral value of u less than q, $u^{(q)}=0$. Beginning with $x=0$ and proceeding outward through a triangular network similar to that employed in § 3, it is possible to determine an A_{qh} with each point of the net. Having determined the A_{qh}'s, it is only necessary to expand and collect the terms of the expression for $f(x)$ and compare with the expression involving the c_{qh}'s to completely determine each c_{qh}.

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* The value of the determinant of the coefficients may be shown to be

$$(v_0 s_0 - u_0 t_0)^{\frac{1}{6}n(n+1)(n+2)} \prod_{a=0}^{n-1} [(n-a)!]^{2(a+1)}.$$

Thus we see that *the most general solution* $f(x) \equiv f(u+iv)$ *of equation* (6) *continuous over the finite complex x-plane is an arbitrary polynomial in u and v of degree n.*

The analytic solution of (6) over the finite complex plane is that special case of the general continuous solution for which

$$f(x) = a_0 + a_1 x + \ldots . + a_k x^k + \ldots . .$$

Replacing x by $u+iv$, it is seen at once that $a_k u^k$ must be zero for $k>n$, and hence $a_k=0$, $k>n$. Moreover, $a_k = c_{k0}$, $k \leqq n$. Hence *the most general analytic solution of* (6) *is an arbitrary polynomial in x of degree n.**

To obtain the most general continuous solution of (6) along any line in the finite complex plane, it is only necessary to observe that by the argument of §§ 3 and 4 it was proved that f is known at a dense set of points on the line if it is known at $n+1$ points of the line which are separated by some convenient unit. The argument at the beginning of this section shows that along any line not parallel to the axis of imaginaries,

$$f(x) = \sum_{j=0}^{n} a_j u^j,$$

where each a is arbitrary, satisfies (6). Since $f(x)$ is continuous, it remains only to show that the a's are determined by the functional values at the $n+1$ points on the line to know that $f(x)$ is the most general continuous solution of (6) along the line. Substitution shows immediately that the a's are uniquely determined by the $n+1$ values of f on the line. Hence *the most general solution of* (6) *continuous along any line not parallel to the axis of imaginaries is an arbitrary polynomial in u of degree n.*

Similarly, *the most general solution of* (6) *continuous along any line not parallel to the axis of reals is an arbitrary polynomial in v of degree n.*

§ 6. *Solutions of the Original Equation.*

The determination of the existence of any solution $f(x)$ of the equation

$$\sum_{i=1}^{n} \gamma_i f(\alpha_i x + \beta_i y) + \gamma_{n+1} f(x) + \gamma_{n+2} f(y) = 0, \tag{1}$$

and the determination of $f(x)$ if it exists, is accomplished by substituting the corresponding solution of the normal equation (6) in (1). It has been shown in § 5 that the general solution of (6) analytic over the finite complex x-plane

* That the analytic solution $f(x)$, if it exists, is a polynomial of degree not greater than n, is readily seen by direct differentiation of (1). If differentiation is made with respect to each of the variables y, $a_j x + \beta_j y$, $j = 1, 2, \ldots, n$ it results that $f^{(k)}(y) = 0$ whenever $k > n$, because the arguments are independent in pairs which are not proportional.

or along any line in the finite complex x-plane is an arbitrary polynomial in x of degree n. Hence, the determination of the general solution of (1) analytic over the finite complex x-plane or along a line in the finite complex x-plane, if it exists, is accomplished by substituting

$$f(x)=c_0+c_1x+\ldots.+c_nx^n$$

in (1). Since the result of this substitution is an identity, the coefficient of any power of the variables must vanish.

It is convenient to consider the terms of degree n independently of the remaining terms. These terms are given by

$$c_n[\sum_{i=1}^{n}\gamma_i(\alpha_ix+\beta_iy)^n+\gamma_{n+1}x^n+\gamma_{n+2}y^n]=0.$$

Now c_n is necessarily zero unless

$$[\sum_{i=1}^{n}\gamma_i(\alpha_ix+\beta_iy)^n+\gamma_{n+1}x^n+\gamma_{n+2}y^n]\equiv 0.$$

Placing the coefficients of this identity equal to zero, we have

$$\sum_{i=1}^{n}\alpha_i^n\gamma_i+\gamma_{n+1}=0,\quad \sum_{i=1}^{n}\alpha_i^{n-k}\beta_i^k\gamma_i=0,\quad \sum_{i=1}^{n}\beta_i^n\gamma_i+\gamma_{n+2}=0,\quad k=1,2,\ldots.,n-1. \tag{10}$$

Since only the ratios of the γ's are significant, it implies no loss of generality to assume $\gamma_{n+2}=-1$. Under this assumption equations (10) may be employed to express the remaining γ's in terms of the α's and β's, provided $f(x)$ contains a non-vanishing term of degree n. If we write $r_i=\alpha_i/\beta_i$, then for $i<n+1$

$$\gamma_i=r_1r_2\ldots.r_n/\beta_i^nr_i\prod_{h=1}^{n}{}'(r_h-r_i),$$

where the prime indicates that h does not take the value i. Solution also gives

$$\gamma_{n+1}=(-1)^nr_1r_2\ldots.r_n.$$

Therefore, if c_nx^n, $c_n\neq 0$, *is a term of the analytic solution* $f(x)$ *of* (1), *it is necessary and sufficient that* (1) *may be written in the form*

$$\sum_{i=1}^{n}\frac{r_1r_2\ldots.r_n}{\beta_i^nr_i\prod_{h=1}^{n}{}'(r_h-r_i)}f(\alpha_ix+\beta_iy)+(-1)^nr_1r_2\ldots.r_nf(x)-f(y)=0,$$

where $r_i=\alpha_i/\beta_i$.

If the solution of (1) includes the term c_mx^m, $m<n$ and $c_m\neq 0$, then

$$\sum_{i=1}^{n}\alpha_i^m\gamma_i+\gamma_{n+1}=0,\quad \sum_{i=1}^{n}\alpha_i^{m-k}\beta_i^k\gamma_i=0,\quad \sum_{i=1}^{n}\beta_i^m\gamma_i+\gamma_{n+2}=0,\qquad k=1,2,\ldots.,m-1.$$

Since there are but $m+1$ linear equations in the γ's, they may be used to

express $m+1$ γ's in terms of the α's, β's and remaining γ's. For the first $m+1$ γ's in terms of the remaining quantities, solution gives

$$\gamma_i=\frac{-\sum_{j=m+2}^{n}\beta_j^m\gamma_j\prod_{h=1}^{m+1}{}'(r_j-r_h)-\gamma_{n+1}+(-1)^{m+1}r_1r_2\ldots r_{i-1}r_{i+1}\ldots r_{m+1}\gamma_{n+2}}{\beta_i^m\prod_{h=1}^{m+1}{}'(r_i-r_h)}, \quad (11)$$

where the prime indicates that h does not take the value i. Hence, *if* c_mx^m, $m<n$ *and* $c_m\neq 0$, *is a term of the analytic solution* $f(x)$ *of* (1), *it is necessary and sufficient that each of the first* $m+1$ γ's *has the value given in* (11).

The computation involved in finding the most general solution of (1) continuous over the entire finite plane is so tedious as to make it expedient to give results only for the general second order equation. The problem for any equation is merely a matter of substitution and algebraic computation. For the second order equation

$$\begin{aligned}\gamma_1 f[(a_{11}+a_{12}i)x+(b_{11}+b_{12}i)y]\\ +\gamma_2 f[(a_{21}+a_{22}i)x+(b_{21}+b_{22}i)y]+\gamma_3 f(x)+\gamma_4 f(y)=0,\end{aligned}$$

where a_{11}, a_{12}, b_{11}, b_{12}, a_{21}, a_{22}, b_{21}, b_{22} are real and $i=\sqrt{-1}$, the normal solution with which substitution must be made is

$$c_{00}+c_{10}u+c_{01}v+c_{20}u^2+c_{11}uv+c_{02}v^2.$$

This substitution shows that c_{00} may be assigned different from zero when and and only when $\gamma_1+\gamma_2+\gamma_3+\gamma_4=0$. It also shows that if we write

$$A_1=\gamma_1a_{11}+\gamma_2a_{21}+\gamma_3,\quad B_1=\gamma_1a_{12}+\gamma_2a_{22},\quad C_1=\gamma_1b_{11}+\gamma_2b_{21}+\gamma_4,\quad D_1=\gamma_1b_{12}+\gamma_2b_{22},$$

then for c_{10} and c_{01} to be independently arbitrary $A_1=B_1=C_1=D_1=0$. However, $c_{10}=c_{01}k_1$ if $A_1=B_1k_1$, $C_1=D_1k_1$, B_1 or $D_1\neq 0$, and $k_1^2=-1$. Furthermore, if we write,

$$\begin{aligned}
&A=\gamma_1a_{11}^2+\gamma_2a_{21}^2+\gamma_3, && B=\gamma_1a_{11}a_{12}+\gamma_2a_{21}a_{22}, && C=\gamma_1a_{12}^2+\gamma_2a_{22}^2,\\
&D=\gamma_1b_{11}^2+\gamma_2b_{21}^2+\gamma_4, && E=\gamma_1b_{11}b_{12}+\gamma_2b_{21}b_{22}, && F=\gamma_1b_{12}^2+\gamma_2b_{22}^2,\\
&G=\gamma_1a_{11}b_{12}+\gamma_2a_{21}b_{22}, && H=\gamma_1a_{11}b_{11}+\gamma_2a_{21}b_{21}, && J=\gamma_1a_{12}b_{12}+\gamma_2a_{22}b_{22},\\
& && K=\gamma_1a_{12}b_{11}+\gamma_2a_{22}b_{21}, &&
\end{aligned}$$

$c_{20}=kc_{02}$ and $c_{11}=mc_{02}$,* then it is easy to show that

(i) $k=-1$ and $m=\pm 2i$ if
  (a) $A+C$, $D+F$, $K-G$, or $H+J\neq 0$,
  (b) B, E, or $J-H\neq 0$,
  (c) $A-C=Bm$, $D-F=Em$, and $2(K+G)=(J-H)m$.

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* The results desired here allow c_{02} to be different from zero, and no attempt is made to discuss the possibilities in case $c_{02}=0$.

(ii) $k=-1$ and $m=0$ if
  (a) A, D, H, or $G \neq 0$,
  (b) B or $E \neq 0$,
  (c) $C=A$, $F=D$, $J=H$, and $G+K=0$.

(iii) $k=-1$ and m is arbitrary if
  (a) A, D, H, or $G \neq 0$,
  (b) $B=E=0$, $C=A$, $F=D$, $J=H$, and $G+K=0$.

(iv) $k=1$ and $m=0$ if
  (a) A^2+B^2, D^2+E^2, or $G^2+H^2 \neq 0$,
  (b) $C=-A$, $F=-D$, $K=G$, and $J=-H$.

The following equations, constructed on the basis of this information, answer interesting questions. The equation

$$3f[(1+2i)x+(3+i)y]-f[(1+3i)x+(6+3i)y]-5f(x)+15f(y)=0$$

has the general continuous solution $f(x)=c_{02}(u^2+v^2)$ showing that *an equation may have a continuous solution, but no non-trivial analytic solution.*

The general solution of

$$(1+i)f[3x+(1-i)y]-3f[x+2y]-(6+9i)f(x)+(10+2i)f(y)=0,$$

continuous over the finite complex x-plane is $c_{20}(u^2+2iuv-v^2)=c_{20}x^2$, showing that *an equation may have its analytic solution as the most general continuous solution.*

The elimination outlined in the footnote of § 2 shows that every solution of an equation of form (1) having no α equal to zero is included in the corresponding solution of the normal equation whose order is equal to the number of distinct ratios β_i/α_i, $i=1, 2, \ldots, n$, in the equation of form (1). One case remains, namely, that in which each ratio β_i/α_i, α_i and β_i different from zero, is equal to at least one other such ratio and at least one α and one β, of different subscripts, are zero. That there are equations of this exceptional type which have infinite series solutions is proved by the equation

$$f(x+y)-f(ix+iy)-f(-ix)-f(-iy)+f(x)+f(y)=0, \quad i=\sqrt{-1},$$

which is satisfied by a series in positive integral powers of x^4 having arbitrary coefficients.

§ 7. *The Converse Theorem.*

It was shown in § 5 that any polynomial in x of degree m_j satisfies the normal equation of order m_j. It will now be proved that any polynomial $p(x)$ in x, of degree m_j, satisfies an equation (1) whose order n is not greater than the number of non-vanishing terms of $p(x)$, plus the sum of the degrees of

such terms, and whose α's, β's and γ_{n+1} and γ_{n+2} may be assigned at will, provided a certain determinant Δ of the α's and β's is not zero as a consequence.

Suppose that

$$p(x)=a_1x^{m_1}+a_2x^{m_2}+\ldots\ldots+a_jx^{m_j}.$$

The substitution of $p(x)$ for $f(x)$ in (1) gives an identity from which

$$\sum_{i=1}^{n}\alpha_i^{m_h}\gamma_i=-\gamma_{n+1},\quad \sum_{i=1}^{n}\alpha_i^{m_h-k}\beta_i^{k}\gamma_i=0,\quad \sum_{i=1}^{n}\beta_i^{m_h}\gamma_i=-\gamma_{n+2}\qquad k=1,2,\ldots\ldots,m-1, \quad (12)$$

for the values $h=1, 2, \ldots\ldots, j$. The total number of independent equations is not greater than $j+\Sigma_{i=1}^{j}m_i$. Setting n equal to the number of independent equations, we have a system of non-homogeneous linear equations in n unknowns $\gamma_1, \gamma_2, \ldots\ldots, \gamma_n$, provided γ_{n+1} and γ_{n+2} are not both assigned equal to zero. A necessary and sufficient condition that the unknown γ's are uniquely determined in terms of the α's, β's and two assigned γ's is that the determinant Δ of the coefficients be different from zero. It is obvious that Δ is a polynomial in the α's and β's. Furthermore, it is at once evident that the term formed by the product of the elements in the principal diagonal is unique. Hence the polynomial in the α's and β's does not vanish identically. Therefore the α's and β's may be assigned in any way such that the polynomial Δ has a value different from zero.

If γ_{n+1} and γ_{n+2} are both assigned equal to zero, equations (12) form a system of n linear homogeneous equations in n unknowns, and the non-vanishing of Δ is a necessary and sufficient condition that each of the unknown γ's is zero. In this case the equation is trivially satisfied by $p(x)$. In general, then, *any polynomial $p(x)$ satisfies an infinity of equations* (1) *whose orders do not exceed the number of non-vanishing terms of $p(x)$ plus the sum of the degrees of such terms and whose α's, β's, γ_{n+1}'s and γ_{n+2}'s may be assigned at will, provided only that a polynomial Δ of the α's and β's does not vanish for the assigned values.*

§ 8. *Discontinuous Solutions.*

Any solution $f(x)$ of an equation (1) satisfies the normal equation whose order n is determined by (1). Suppose that the domain of $f(x)$ is any line in the finite complex x-plane, and that $f(x)$ is continuous in some interval of length $\delta>0$ of the line. It is readily seen from the normal equation satisfied by f that if the first $n+1$ arguments are so chosen that they represent points in the interval while the remaining argument represents a point outside the interval, $f(x)$ is determined at the last-named point as the sum of continuous functions. Therefore f is continuous at the outside point. In this way f may be shown to be continuous at all points in the two intervals of length

δ/n which lie at the ends of the given interval. Therefore f is continuous in an interval of length $[(n+2)\delta]/n$. Any finite interval of length σ may be reached in this manner by a finite number of extensions of the interval of length δ. We may therefore state that *if $f(x)$ has a finite point of discontinuity on any line in the complex x-plane, it has a point of discontinuity in every interval of the line, however small.*

Suppose $f(x)$ is continuous in a region of the finite complex x-plane. A circle may be inscribed in the region such that $f(x)$ is continuous in the closed region of which the circle is the boundary. Suppose the radius of this circle is δ and consider a concentric circle of radius $[(n+1)\delta]/n$. By means of the normal equation f may be determined at any point in the area between the circles as the sum of $n+1$ continuous functions, namely, f at $n+1$ points in the circle of radius δ. Therefore f at the point between the circles is continuous. This is true for every point of the area between the circles, and hence f is continuous in the circle of radius $[(n+1)\delta]/n$. This process may obviously be repeated to prove that f is continuous in any finite region of the plane. Hence, *if $f(x)$ has a point of discontinuity in the finite complex x-plane, it has a point of discontinuity in every finite region of the plane.*

G. Hamel (*loc. cit.*) has exhibited a discontinuous solution* $f(x)$ of the Cauchy equation

$$f(x+y)=f(x)+f(y).$$

From the treatment in § 2 it is clear that f also satisfies

$$f(2x+y)-2f(x+y)+f(y)=0.$$

Replacing y by $hx+y$, multiplying the equation by $(-1)^h(n-1)!/h!(n-h-1)!$ for successive values $h=0,1,\ldots,n-1$, and adding the n equations so formed, it is easily seen that

$$\sum_{k=0}^{n+1}(-1)^k\frac{(n+1)!}{k!(n+1-k)!}f(kx+y)=0. \tag{6}$$

We therefore have a discontinuous solution of the normal equation for each order n.

§ 9. *Certain Types of Equations having Variable Coefficients.*

The functional equations that have been discussed may be employed to solve certain equations of the form,

$$\sum_{i=1}^{n}\phi_i(x,y)f(\alpha_i x+\beta_i y)+\phi_{n+1}(x,y)f(x)+\phi_{n+2}(x,y)f(y)+\phi_{n+3}(x,y)=0, \tag{13}$$

where the ϕ's are known functions. A general statement and a few examples suffice to indicate some of the equations that may be solved. Suppose there are k transformations

$$x=\gamma_j x'+\delta_j y', \quad y=\lambda_j x'+\mu_j y',$$

* This solution is obtained on the assumption of the validity of the Zermelo axiom.

which may be applied to (13) to obtain new equations such that if each equation is multiplied by a non-zero constant, the sum of them is of form (1). A solution of (13), if it exists, is included in the corresponding solution of the auxiliary equation of form (1) deduced from (13). In order to find a solution of (13) it is sufficient to substitute the solution of the auxiliary equation and compute the coefficients of the variables.

Equation (13) includes the non-homogeneous equation in which $\phi_i(x, y)$, $i=1, 2, \ldots, n+2$, is further restricted to be a constant. In this case it is obvious from equation (13) that $\phi_{n+3}(x, y)$ is a polynomial in x and y if an analytic solution exists, and a polynomial in u, v, s and t if a continuous solution exists. Such a non-homogeneous equation is*

$$f(x+y)=f(x)+f(y)+2xy. \tag{14}$$

Transformations which may be used to solve this equation are

$$x=x', \ y=x'-y' \ \text{ and } \ x=x', \ y=y'-x'$$

whence, after dropping the primes,

$$f(2x-y)-f(x-y)-f(-x+y)-2f(x)+f(y)=0.$$

The arguments $x-y$ and $-x+y$ are proportional and the normal equation is therefore of order 2. Hence the general solution of (14), analytic over the finite complex x-plane is readily seen to be $f(x)=a_1x+x^2$, where a_1 is arbitrary. The general solution of (14) continuous over the finite complex x-plane is $f(x)=a_{10}u+a_{01}v+x^2$, where a_{10} and a_{01} are arbitrary.

Suppose all the ϕ's are constant and $\phi_{n+3}\neq 0$. If $\Sigma_{i=1}^{n+2}\phi_i\neq 0$, then $f(0)$ is finite and uniquely determined, and the transformation $f(x)=g(x)+f(0)$ may be employed to obtain an equation (1) of order n in $g(x)$. Hence $f(x)$ is a polynomial (in x, in u and v, in u or in v, as the case may be) of degree not greater than n. The transformation used in this case has the advantage of furnishing an auxiliary equation (1) whose order is not greater than that of the original equation (13).

An equation which illustrates reduction by interchanging arguments of f is

$$(\cos^2 x)f(x+y)+(\sin^2 x)f(x-y)-f(x)-f(y)-2(\cos 2x)xy=0.$$

If y is replaced by $-y$ the equation becomes

$$(\sin^2 x)f(x+y)+(\cos^2 x)f(x-y)-f(x)-f(-y)+2(\cos 2x)xy=0.$$

The sum of these equations is of form (1). The solution analytic over the finite complex x-plane of the equation having variable coefficients is $f(x)=x^2$. It is easily seen that this is the most general continuous solution.

The equation

$$2f(2x+y)+(x+y)f(x-y)+3f(x)-3f(y)=0$$

* *American Mathematical Monthly*, Vol. XXIV (1917), p. 178.

may be reduced by replacing x by $2x+y$ and y by $x+2y$, whence

$$2f(5x+4y)+3(x+y)f(x-y)+3f(2x+y)-3f(x+2y)=0,$$

and subtracting 3 times the original equation from the transformed equation. The reduced equation is

$$2f(5x+4y)-3f(2x+y)-3f(x+2y)-9f(x)+9f(y)=0.$$

It is readily seen that the equation with variable coefficient has no continuous solution.

If one of the first $n+2$ ϕ's of (13) is variable while the remaining ϕ's are constant the product of that ϕ and the corresponding $f(\alpha_i x+\beta_i y)$ is a polynomial. The variable ϕ is therefore a rational function whose denominator is a factor of $f(\alpha_i x+\beta_i y)$. An equation illustrating this point is

$$2\left[\frac{x+y}{x-y}\right]^2 f(x-y)-f(x+2y)-f(x)+2f(y)=0.$$

The equation obtained by replacing x by $2x+y$ and y by $x+2y$ is

$$18\left[\frac{x+y}{x-y}\right]^2 f(x-y)-f(4x+5y)-f(2x+y)+2f(x+2y)=0.$$

The equation obtained by subtracting the transformed equation from 9 times the original one gives an equation (1) of order 3. The analytic solution $f(x)$ of the equation with a variable coefficient is then easily seen to be ax^2, where a is arbitrary.

§ 10. *Application to Binomial Equations.*

An application of linear functional equations having constant coefficients may also be made to certain equations of the form,

$$\prod_{i=1}^{k}[f(\alpha_i x+\beta_i y)]^{\gamma_i}=C\prod_{i=k+1}^{n+1}[f(\alpha_i x+\beta_i y)]^{\gamma_i}[f(y)]^{\gamma_{n+2}}, \tag{15}$$

where C is a constant, the real part of each γ is positive and no α is zero. Let us first consider the solution $f(x)$ of (15) analytic at all points in the finite complex plane. Suppose that $f(x)$ has a zero at some point $x=a$. Let $y=a$ in (15). Then

$$\prod_{i=1}^{k}[f(\alpha_i x+\beta_i a)]^{\gamma_i}=0$$

for all values of x. Hence there is a finite region in which $f(x)$ has an infinity of zeros. But this is impossible since $f(x)$ is analytic throughout the finite plane. Therefore, *when $f(x)$ is analytic throughout the finite complex plane, and not identically zero, it is never zero.* The case of a continuous solution $f(x)$ of (15) presents more difficulty. It is evident that *if one member of* (15) *has either no factor or only one factor involving f, then $f(x)$ is never zero unless it is identically so.*

The function $\phi(x)=\log f(x)$, where it is understood that the principal determination of the logarithm is employed, is analytic or continuous with f when the latter has no zeros. Therefore, in each of the cases considered above, $\phi(x)$ satisfies the equation

$$\sum_{i=1}^{k}\gamma_i\phi(\alpha_i x+\beta_i y)-\sum_{i=k+1}^{n+1}\gamma_i\phi(\alpha_i x+\beta_i y)-\gamma_{n+2}\phi(y)-K+2s\pi i=0,$$

where K is the principal determination of $\log C$ and s is any integer. For any given s $\phi(x)$ is a polynomial of degree not greater than n. Furthermore, the argumentation of § 9 shows that a variation in s affects only the constant term of $\phi(x)$. Therefore, *in each of the cases considered above, $f(x)$ is an exponential function of the form*

$$f(x)=e^{P+k(s)}, \tag{16}$$

where P is a polynomial of degree not greater than n, and $k(s)$ is a constant depending on s. Thus we see that in general the solutions of (15) are given by (16) for the various possible values of $k(s)$.

The same result may be stated for the continuous solution $f(x)$ of (15), *when each member involves at least two factors containing f, provided $f(0)\neq 0$.* For $\phi(x)$ as defined, is continuous in some region about the point $x=0$ since $f(x)$ is different from zero in some such region. For a given s, therefore, $\phi(x)$ must be everywhere continuous in the finite complex plane because it satisfies a non-exceptional equation of type (1).

The equation,

$$\psi(x+y)\psi(x-y)=[\psi(x)\psi(y)]^2,$$

mentioned in § 1, is included in the last case considered. For suppose there is at least one point $x=b$ at which $\psi(x)$ is not zero. Let $x=y=b$. Then

$$\psi(2b)\psi(0)=[\psi(b)]^4\neq 0,$$

and $\psi(0)\neq 0$. It is easily seen that the solutions $\psi(x)$ analytic over the finite complex plane are

$$\psi(x)=e^{ax^2-s\pi i}=\pm e^{ax^2},$$

and the solutions $\psi(x)$ continuous over the finite complex plane are

$$\psi(x)=e^{c_{20}u^2+c_{11}uv+c_{02}v^2-s\pi i}=\pm e^{c_{20}u^2+c_{11}uv+c_{02}v^2},$$

where a and the c's are arbitrary constants.

§ 11. *Equations Involving More than One Function.*

Consider the equation

$$\sum_{i=1}^{n}\gamma_i f_i(\alpha_i x+\beta_i y)+\gamma_{n+1}f_{n+1}(x)+\gamma_{n+2}f_{n+2}(y)=0, \tag{17}$$

where no γ is zero and the f's are unknown, continuous, single-valued functions to be determined if possible so that (17) shall be identically satisfied by them. The functions f_i may or may not all be distinct. The method of elimination employed in § 2 is applicable to (17). If no α is zero it is evident then that f_{n+2} satisfies the normal equation of order n. Therefore f_{n+2} is a polynomial of degree not greater than n. Under the assumption that no α_i and no β_i are zero, and no two of the ratios β_i/α_i are equal, any term may be given the argument y by a linear transformation which makes no α and no β zero. In this case, therefore, *every function f_i of* (17) *is a polynomial of degree not greater than n.* To find any necessary restrictions on the coefficients of these polynomials, it is sufficient to substitute the n-th degree polynomials having general coefficients in (17), and to equate to zero the resulting coefficients of the variables. It is obvious that equation (1) is a special case of (17).

If some of the ratios β_i/α_i are equal it may be assumed without loss of generality that equation (17) is so arranged that functions having arguments of a common ratio are placed consecutively. If all the functions having arguments of a common ratio have subscripts i, such that $g \overline{\gtrless} i \overline{\gtrless} h$, then we may write

$$F_h(\alpha_h x+\beta_h y)=\sum_{i=g}^{h}\gamma_i f_i(\alpha_i x+\beta_i y).$$

Equation (17) may now be written

$$\Sigma F_h(\alpha_h x+\beta_h y)+F_{n+1}(x)+F_{n+2}(y)=0, \tag{18}$$

where no α and no β are zero, and no two ratios β_h/α_h are equal. We denote by $q+2$ the number of terms in the first member of (18). Each F is a polynomial of degree not greater than q. Each F therefore determines a non-homogeneous equation in certain f's having arguments differing by constant factors. If the f's of any F are identical, that is, if the f's of F are the same function, F determines a non-homogeneous mixed q-difference equation satisfied by f. The equations of type (1), which have no α equal to zero, but have some ratios β_i/α_i equal, or some β's zero, are special cases of (18) given by $F_{n+2}(y)=\gamma_{n+2}f(y)$. The equations of the exceptional case noted in the last paragraph of § 6 are equations of form (18) which have no F a constant multiple of a single f. In this connection it is interesting to note that the function f of the example in the paragraph cited satisfies the equations

$$f(x)-f(-ix)=f(x)-f(ix)=0,$$

whence

$$f(-ix)=f(ix)=f(-x)=f(x).$$